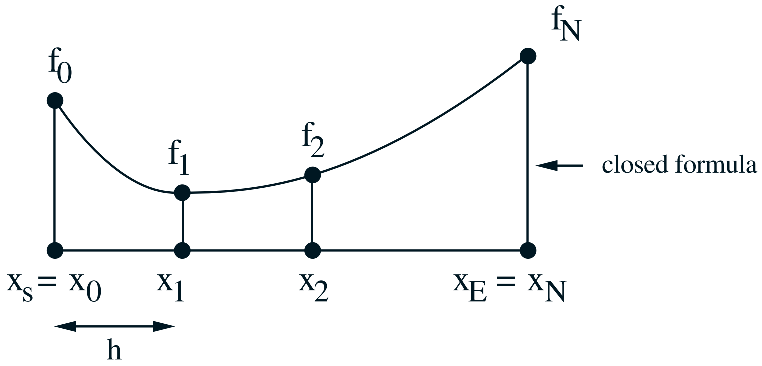
GAUSS QUADRATURE

- In general for Newton-Cotes (equispaced interpolation points/ data points/ integration points/ nodes).

$$\int_{x_S}^{x_E} f(x)dx = h[w'_o f_o + w'_1 f_1 + \ldots + w'_N f_N] + E$$



- Note that for Newton-Cotes formulae only the weighting coefficients w_i were unknown and the x_i were fixed

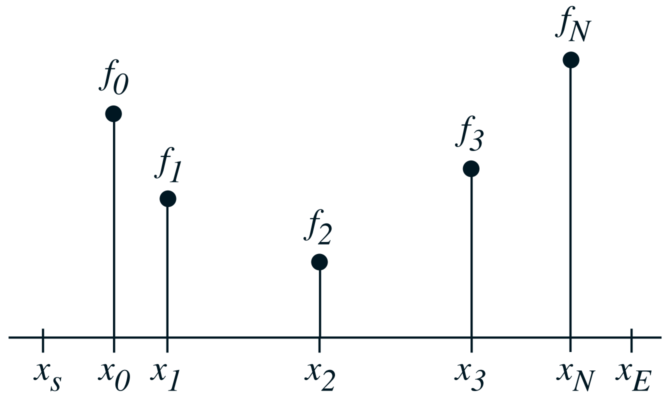
- However the number of and placement of the integration points influences the accuracy of the Newton-Cotes formulae:
  - N even $\rightarrow N^{th}$ degree interpolation function exactly integrates an $N+1^{th}$ degree polynomial $\rightarrow$ This is due to the placement of one of the data points.
  - N odd $\rightarrow N^{th}$ degree interpolation function exactly integrates an N^{th} degree polynomial.

- ***Concept: Let's allow the placement of the integration points to vary such that we further increase the degree of the polynomial we can integrate exactly for a given number of integration points.***

- ***In fact we can integrate an*** $2N+1$ ***degree polynomial exactly with only*** $N+1$ ***integration points***

- Assume that for Gauss Quadrature the form of the integration rule is

$$\int_{x_S}^{x_E} f(x)dx = [w_o f_o + w_1 f_1 + \ldots + w_N f_N] + E$$



- In ***deriving*** (not applying) these integration formulae
  - Location of the integration points, $x_i \quad i = O, N$ are unknown
  - Integration formulae weights, $w_i \quad i = O, N$ are unknown
- $2(N+1)$ unknowns $\rightarrow$ we will be able to exactly integrate any $2N+1$ degree polynomial!

Derivation of Gauss Quadrature by Integrating Exact Polynomials and Matching

Derive 1 point Gauss-Quadrature

- 2 unknowns w_o, x_o which will exactly integrate any linear function

- Let the ***general*** polynomial be

$$f(x) = Ax + B$$

where the coefficients A, B can equal any value

- Also consider the integration interval to be $[-1, +1]$ such that $x_S = -1$ and $x_E = +1$ (no loss in generality since we can always transform coordinates).

$$\int_{-1}^{+1} f(x)dx = w_o f(x_o)$$

- Substituting in the form of $f(x)$

$$\int_{-1}^{+1} (Ax + B)dx = w_o(Ax_o + B) \quad \Rightarrow$$

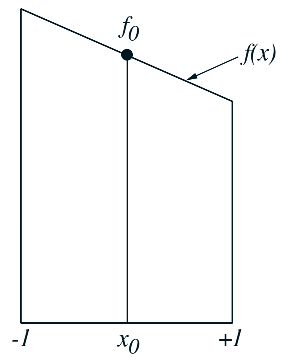
$$\left[A\frac{x^2}{2} + Bx\right]_{-1}^{+1} = w_o(Ax_o + B) \quad \Rightarrow$$

$$A(0) + B(2) = A(x_o w_o) + B(w_o)$$

- In order for this to be true for any 1st degree polynomial (i.e. any A and B).

$$\begin{cases} 0 = x_o w_o \\ 2 = w_o \end{cases}$$

- Therefore $x_o = 0$, $w_o = 2$ for 1 point $(N = 1)$ Gauss Quadrature.



- We can integrate exactly with only 1 point for a linear function while for Newton-Cotes we needed two points!

Derive a 2 point Gauss Quadrature Formula

- The general form of the integration formula is

$$I = w_o f_o + w_1 f_1$$

- w_o, x_o, w_1, x_1 are all unknowns

- 4 unknowns $\Rightarrow$ we can fit a 3rd degree polynomial exactly

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

- Substituting in for $f(x)$ into the general form of the integration rule

$$\int_{-1}^{+1} f(x)dx = w_o f(x_o) + w_1 f(x_1)$$

$$\Rightarrow$$

$$\int_{-1}^{+1} [Ax^3 + Bx^2 + Cx + D]dx = w_o[Ax_o^3 + Bx_o^2 + Cx_o + D] + w_1[Ax_1^3 + Bx_1^2 + Cx_1 + D]$$

$$\Rightarrow$$

$$\left[\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx\right]_{-1}^{+1} = w_o(Ax_o^3 + Bx_o^2 + Cx_o + D) + w_1(Ax_1^3 + Bx_1^2 + Cx_1 + D)$$

$$\Rightarrow$$

$$A[w_o x_o^3 + w_1 x_1^3] + B\left[w_o x_o^2 + w_1 x_1^2 - \frac{2}{3}\right] + C[w_o x_o + w_1 x_1] + D[w_o + w_1 - 2] = 0$$

- In order for this to be true for ***any*** third degree polynomial (i.e. all arbitrary coefficients, A, B, C, D), we must have:

$$w_o x_o^3 + w_1 x_1^3 = 0$$

$$w_o x_o^2 + w_1 x_1^2 - \frac{2}{3} = 0$$

$$w_o x_o + w_1 x_1 = 0$$

$$w_o + w_1 - 2 = 0$$

- 4 nonlinear equations → 4 unknowns

$$w_o = 1 \quad \text{and} \quad w_1 = 1$$

$$x_o = -\sqrt{\frac{1}{3}} \quad \text{and} \quad x_1 = +\sqrt{\frac{1}{3}}$$

- All polynomials of degree 3 or less will be ***exactly*** integrated with a Gauss-Legendre 2 point formula.

Gauss Legendre Formulae

$$I = \int_{-1}^{+1} f(x)dx = \sum_{i=0}^{N} w_i f_i + E$$

N	$N+1$	x_i, $i = 0, N$	w_i	Exact for polynomials of degree
0	1	0	2	1
1	2	$-\sqrt{\frac{1}{3}}, +\sqrt{\frac{1}{3}}$	1, 1	3
2	3	-0.774597, 0, +0.774597	0.5555, 0.8889, 0.5555	5
N	$N+1$			$2N+1$

N	$N + 1$	x_i, $i = 0, N$	w_i	Exact for polynomials of degree
3	4	-0.86113631	0.34785485	7
		-0.33998104	0.65214515	
		0.33998104	0.65214515	
		0.86113631	0.34785485	
4	5	-0.90617985	0.23692689	9
		-0.53846931	0.47862867	
		0.00000000	0.56888889	
		0.53846931	0.47862867	
		0.90617985	0.23692689	
5	6	-0.93246951	0.17132449	11
		-0.66120939	0.36076157	
		-0.23861919	0.46791393	
		0.23861919	0.46791393	
		0.66120939	0.36076157	
		0.93246951	0.17132449	

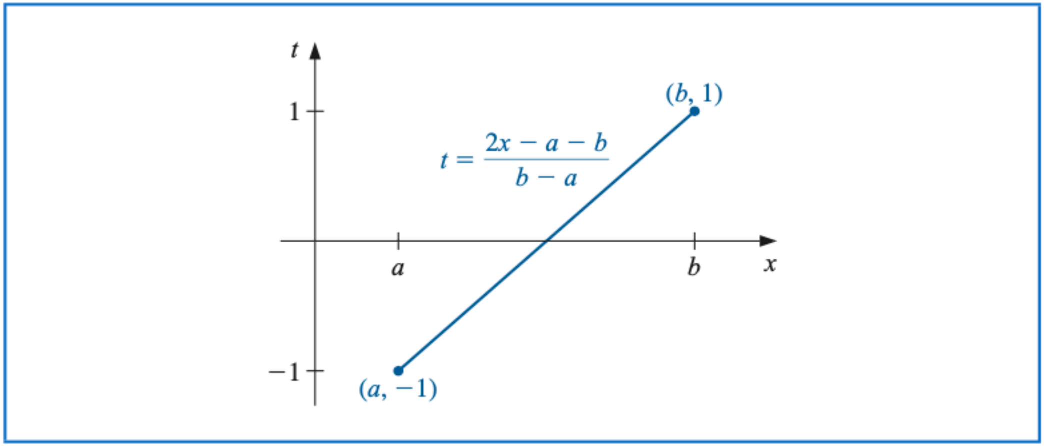
- Notes
  - $N+1$ = the number of integration points
  - Integration points are symmetrical on $[-1, +1]$
  - Formulae can be applied on any interval using a coordinate transformation
  - $N+1$ integration points $\rightarrow$ will integrate polynomials of up to degree $2N+1$ exactly.
    - Recall that Newton Cotes $\rightarrow$ $N+1$ integration points only integrates an $N^{th}/N+1^{th}$ degree polynomial exactly depending on N being odd or even.
    - For Gauss-Legendre integration, we allowed both weights and integration point locations to vary to match an integral exactly $\Rightarrow$ more d.o.f. $\Rightarrow$ allows you to match a higher degree polynomial!
    - An alternative way of looking at Gauss-Legendre integration formulae is that we use Hermite interpolation instead of Lagrange interpolation! (How can this be since Hermite interpolation involves derivatives $\rightarrow$ let's examine this!)

Gaussian Quadrature on Arbitrary Intervals

An integral $\int_a^b f(x)\,dx$ over an arbitrary $[a, b]$ can be transformed into an integral over $[-1, 1]$ by using the change of variables (see Figure 4.17):

$$t = \frac{2x - a - b}{b - a} \Longleftrightarrow x = \frac{1}{2}[(b - a)t + a + b].$$

Figure



This permits Gaussian quadrature to be applied to any interval $[a, b]$, because

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{(b - a)t + (b + a)}{2}\right)\frac{(b - a)}{2}\,dt.$$

Examples (2-point Gauss-Legendre integration)

Q1) Evaluate the integral $I = \dfrac{1}{2}\displaystyle\int_{-1}^{1} e^{-(1+x)^2/4} dx$

Soln. Since the 2-point Gauss-Legendre formula yields:

$$\int_{-1}^{1} f(x)dx = 1 * f\left(\frac{1}{\sqrt{3}}\right) + 1 * f\left(-\frac{1}{\sqrt{3}}\right)$$

We have $I = \dfrac{1}{2}\left[e^{-\left(\frac{1+\frac{1}{\sqrt{3}}}{2}\right)^2} + e^{-\left(\frac{1-\frac{1}{\sqrt{3}}}{2}\right)^2}\right]$ = 0.746594688

The true solution is $0.7468241328.....$

Q2) Evaluate the integral $I = \displaystyle\int_{-1}^{1} \frac{1}{2+x} dx$

Soln.: Using the 2-point Gauss-Legendre formula gives

$I = \dfrac{1}{2+\frac{1}{\sqrt{3}}} + \dfrac{1}{2-\frac{1}{\sqrt{3}}} \approx 1.0909090909....$ where as the true solution is

$I = \log 3 - \log 1 = 1.09861228866811$

Homework:

Q3) Evaluate $I = \displaystyle\int_{1}^{3} x^6 - x^2 \sin(2x) dx$ using the 2-point and 3-point Gauss-Legendre formula.

Hint: Make sure to use the transformation of coordinates to change the limits of integration from $\displaystyle\int_{1}^{3} (\cdot) dx$ *to* $\displaystyle\int_{-1}^{1} (\cdot) dx$