# Time Series Models: Moving Average model

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## Moving Average model of order 1

**MA(1)**:
Let $\varepsilon_t$ be white noise. The MA(1) model is defined as follows:

$$Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1}; \quad \mu, \theta \text{ are constants.}$$

Why is this model order 1?

## Properties of MA(1): Auto-covariances

1. $E[Y_t] = \mu + 0 + 0 = \mu$ is independent of time,
2. $$\begin{aligned} Var(Y_t) &= Var(\mu) + Var(\varepsilon_t) + \theta^2 Var(\varepsilon_{t-1}) \\ &= 0 + \sigma^2 + \theta^2\sigma^2 \\ &= (1+\theta^2)\sigma^2 \text{ is independent of time,} \end{aligned}$$
3. First auto-covariance:
$$\begin{aligned} \gamma_{1t} &:= E(Y_t - \mu)(Y_{t-j} - \mu) = E(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2}) \\ &= \theta\sigma^2 \quad \text{b/c } E[\varepsilon_t\varepsilon_{t-\tau}] = 0 \ \forall t \neq \tau. \end{aligned}$$
4. $\gamma_{jt} = 0 \quad \forall j \geq 2.$

## Properties of MA(1): Stationarity and ergodicity

- For MA(1), $\mu_t, \gamma_{jt}$ are constants and hence independent of $t$. Further, $\gamma_j$ is symmetric by definition.
  Therefore, $MA(1)$ **is a stationary process.**

- $\sum_{j \geq 0} |\gamma_j| = |\gamma_0| + |\gamma_1| + \sum_{j \geq 2} |\gamma_j| = (1 + \theta^2)\sigma^2 + \theta\sigma^2 < \infty$.
  **This implies MA(1) is an ergodic process.**

## Auto-correlation

$\rho_j := \frac{\gamma_j}{\gamma_0}$ is the auto-correlation function.

$|\rho_j| \leq 1 \quad \forall j$ by Cauchy-Schwartz inequality and $\rho_0 = 1$ always.

For MA(1): $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\theta}{1+\theta^2}$, $\rho_j = 0 \quad \forall j > 1$.

## MA(q) process

Definition: $Y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1} + ... + \theta_q\varepsilon_{t-q}$ where $\{\varepsilon_t\}$ is a white noise process and $(\theta_1, ..., \theta_q)$ are real constants.

**Exercise**: $E(Y_t) = \mu$, $\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + ... + \theta_q^2)\sigma^2$ and $\gamma_j = 0 \quad \forall j > q$ which implies MA(q) is also **stationary** and **ergodic.**